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Recent astrophysical observations suggest that the value of fine structure constant $\alpha = e^2/\hbar c$ may be slowly increasing with time. This may be due to an increase of e or a decrease of c , or both. In this article, we argue from model independent considerations that this variation should be considered adiabatic. Then, we examine in detail the consequences of such an adiabatic variation in the context of a specific model of quantized charged black holes. We find that the second law of black hole thermodynamics is obeyed, regardless of the origin of the variation, and that interesting constraints arise on the charge and mass of black holes. Finally, we estimate the work done on a black hole of mass M due to the proposed α variation.

I. INTRODUCTION

Some recent astrophysical observations suggest that, α , the fine structure constant of quantum electrodynamics is not a constant, but has increased by approximately $\delta\alpha/\alpha \approx 10^{-5}$ over 6 – 10 billion years, corresponding to a rate of change of about $\dot{\alpha}/\alpha \approx 10^{-16}/\text{year}$ [1,2]. Although this result is unconfirmed and quite controversial (see for example the recent paper by Bahcall et al [3] where upper bounds on the variation of α and its rate of variation were set at $\delta\alpha/\alpha < 10^{-4}$ over about 10 Billion years and $\dot{\alpha}/\alpha < 10^{-13}/\text{year}$), it provides motivation for learning as much as we can about the consequences of such potential variations (for other observations see [4], a review [5], implications of varying α in cosmology [6], in string and brane theory [7] and other probable consequences [8]).

Now, from the definition

$$\alpha = \frac{e^2}{\hbar c} \quad (1)$$

it is clear that one of the following may be true: (i) the electronic charge e is increasing (ii) the speed of light c is decreasing (iii) Planck's constant \hbar is decreasing. Also, any combination of the above is possible, and moreover, situations such as *increasing* \hbar and decreasing c (at a faster rate) are also viable. Although it was suggested by Duff in [9] that only dimensionless constants can legitimately be thought to evolve, here we adopt the viewpoint of Carlip [10], whereby varying c , for example, is simply regarded as a shorthand for variation of all dimensionless parameters that depend on c . Moreover, as will be seen in subsequent sections, our final results depend on the (dimensionless) quantities α and $\delta\alpha/\alpha$.

It was argued in [11] that since the Bekenstein-Hawking entropy of a Reissner-Nordström black hole is given in terms of its mass M , charge Q and the fundamental constants of nature by:

$$S_{BH} = \frac{k_B \pi G}{\hbar c} \left[M + \sqrt{M^2 - \frac{Q^2}{G}} \right]^2, \quad (2)$$

increase of Q appears to decrease S_{BH} , while decrease of c or \hbar increases S_{BH} . Since only the latter is consistent with the second law of black hole thermodynamics, the authors of [11] concluded that the variation of α must be due to a decrease in c (here and in other articles, for simplicity it is assumed that \hbar does not change). However, it was recently shown in [12] that the above conclusion may be somewhat premature. If one rigorously defines a thermodynamic ensemble by considering a black hole in a 'box', then S_{BH} increases whenever α increases, irrespective of whether the change is due to an increase of e or decrease of c [12,13]. This means that the observed variation of fine structure constant, and the second law of black hole thermodynamics are perfectly compatible with each other, without further assumptions. In fact, one might even say that if α changes at all, it has to increase, if black hole thermodynamics has to hold. The issue of compatibility of varying α with the second law was also examined in [14].

In this article, we take a somewhat different approach by first arguing that on fairly general grounds that the variation of α should be treated as adiabatic in the context of macroscopic, isolated black holes. For concreteness, we then focus on a specific model of quantized spherically symmetric charged black holes, considered in [15,16], and show that an increase in α automatically implies an increase in S_{BH} , once again, irrespective of the source of the variation, and consistent with the conclusions of [12]. We remind the reader that the model considered here as well as the related spectrum of black hole observables are somewhat speculative. However, as was shown in [10], the qualitative features of our results may be expected to survive for spectra arising in quantum theories of gravity such as string theory and loop quantum gravity.

There are two essential ingredients associated with our analysis, namely:

- Black holes are isolated dynamical systems that only interact with the outside world through the time variation of fundamental constants.
- The α variation is adiabatic (which will be proven in the next section)

Under the above assumptions, there are certain quantum numbers that do not change under the variation. For example, it was first argued by Bekenstein [17] that black hole entropy, or horizon area, is an adiabatic invariant, and that its quantum spectrum should be equally space [18]. This had been assumed in the algebraic approach to black hole area quantization by Bekenstein and Gour [19]. Subsequently it was shown in [15] that for charged black holes, it is actually the “entropy above extremality” which is an adiabatic invariant. Thus the latter does not change under slow variation of α . However, the total entropy does change and in fact increases with increasing α . This will be shown in section (III). Another natural consequence of these assumptions is that, as with most adiabatic variations in quantum mechanics, there will be a shift in energy of the black hole. This does not, of course, violate any conservation laws, since the energy is provided by ‘work’ being done by external forces to change the constants.

In the next section, we argue on general grounds that for black holes which are far from extremality, the observed α -variation can indeed be regarded as adiabatic. This result is *independent* of any specific model of black hole. In subsequent sections we examine the consequences of the adiabatic invariant in the context of the specific quantum model for charged black holes mentioned above. Finally, we conclude with a summary and some open questions.

II. ADIABATICITY OF VARIATION OF FINE STRUCTURE CONSTANT

First let us consider uncharged black holes, which have associated with them a natural time scale

$$t_{char} = \frac{2GM}{c^3} . \quad (3)$$

This scale can either be thought of as the characteristic light transition time across one horizon radius, or as ‘inverse-Hawking temperature’ $t_{char} = \hbar/4\pi k_B T_H$. Suppose $H(q, p; \lambda)$ is the Hamiltonian governing the dynamics of these black holes with λ any slowly varying parameter. In the current context, it could be α , c or e . A necessary condition for adiabaticity is that the parameter change only a little during the characteristic time scale of the system. More precisely, [20]:

$$\frac{\dot{\lambda}}{\lambda} \ll \frac{1}{t_{char}} , \quad (4)$$

Plugging in the experimental value $\dot{\alpha}/\alpha \approx 10^{-16}/year = 10^{-23}/sec$ [1], we get from (3) and (4) :

$$\frac{c^3}{8\pi GM} \gg 10^{-9} ,$$

which yields the bound

$$M \ll 10^{27} M_{\odot} , \quad (5)$$

where $M_{\odot} = 2 \times 10^{33} g$ is the solar mass. If the upper bound of Bahcall et al was used instead [3], namely $\dot{\alpha}/\alpha < 10^{-13}/year$, then the upper bound on M would have been $10^{24} M_{\odot}$. In any case, since masses of astrophysical black holes are at most of the order of billion solar masses, the above bound is always satisfied, and the variation of α can be regarded as adiabatic as far as phenomena involving black holes are concerned. Adiabatic variations of α were also considered recently in [21].

The analysis can easily be extended to black holes with charge, in which case,

$$t_{char} = \frac{\hbar}{4\pi k_B T_H} = \frac{[GM + \sqrt{(GM)^2 - GQ^2}]^2}{2c^3 \sqrt{(GM)^2 - GQ^2}} = \frac{2GM}{c^3} \left[1 + \frac{1}{4} \left(\frac{Q}{\sqrt{GM}} \right)^4 + \dots \right] \quad (6)$$

Since charge appears as a fourth-order effect, its influence can be ignored for all practical purposes in the adiabaticity analysis (as long as the black hole is away from extremality). Similar conclusions would follow for black holes with angular momentum, with the replacement $Q \rightarrow Jc/\sqrt{GM}$. Thus, the bound (5) appears to be quite robust.

The above analysis is useful if one knows the adiabatic invariants associated with black holes. Bekenstein has long argued that the entropy of a black hole is an adiabatic invariant [17]. We will now outline a proof that this is so for uncharged black holes. Our basic assumption is that the time scale t_{char} is due to an intrinsic periodicity of the system, with characteristic frequency:

$$\omega(E) \propto \frac{c^3}{2GM} \quad , \quad (7)$$

where $E = Mc^2$. Since black holes are static, it is at first glance difficult to see what the physical source of this periodicity might be. Two possibilities have been suggested in the literature. This intrinsic periodicity can be connected with the periodicity in Euclidean time that characterizes the thermodynamical properties of the black hole [15], in which case the constant of proportionality is $1/8$. Alternatively, one can use the high damping limit of the quasi-normal mode frequency of the black hole [22–24] in which case the constant of proportionality is $\ln(3)/4\pi$.

It can be shown that for any periodic system, there exists an adiabatic invariant, which can be calculated (up to a constant shift) as follows:

$$\mathcal{J} \equiv \frac{1}{2\pi} \oint pdq \propto \int \frac{dE}{\omega(E)} \quad , \quad (8)$$

where (q, p) are its phase space variables. Using (7), we get:

$$\mathcal{J} \propto \frac{8GM^2}{c} = \frac{\hbar S_{BH}}{2\pi k_B} \quad , \quad (9)$$

where S_{BH} is the Bekenstein-Hawking entropy of the Schwarzschild black hole. Invoking the Bohr-Sommerfeld quantisation principle yields the result that black hole entropy is uniformly spaced:

$$S_{BH} \propto 2n\pi k_B \quad , \quad n = 0, 1, 2, \dots \quad (10)$$

From the adiabatic theorem of quantum mechanics, under adiabatic variation of any parameter in the entropy, the quantum number n does not change [25]. We will return to the issue of adiabatic invariance for charged black holes in the next section.

III. CHARGED BLACK HOLES

To further probe the consequences of such adiabatic variation of parameters, we now consider a specific reduced phase space model of charged black hole [15]. The starting point of this model is the result of [26,27] that the dynamics of static spherically symmetric charged configurations in *any* classical theory of gravity in d -spacetime dimensions is governed by an effective action of the form

$$I = \int dt \left(P_M \dot{M} c^2 + P_Q \dot{Q} - H(M, Q) \right) \quad (11)$$

where M and Q are the mass and the charge respectively and P_M, P_Q the corresponding conjugate momenta. The momentum P_M has the interpretation of asymptotic time difference between the left and right wedges of a Kruskal diagram. Note that H is independent of P_M and P_Q , such that from Hamilton's equations, M and Q are constants of motion.

Now, to incorporate thermodynamics of black holes, one assumes that the conjugate momentum P_M is periodic with period equal to inverse Hawking temperature (times \hbar). That is,

$$P_M \sim P_M + \frac{\hbar}{k_B T_H(M, Q)} \quad . \quad (12)$$

Similar assumptions were made in the past using different arguments [28]. Note that the above identification implies that the (M, P_M) phase subspace has a wedge removed from it, which makes it difficult, if not impossible to quantise on the full phase-space. Thus, one can make a canonical transformation $(M, Q, P_M, P_Q) \rightarrow (X, Q, \Pi_X, \Pi_Q)$, which on the one hand ‘opens up’ the phase space, and on the other hand, naturally incorporates the periodicity (12) [26]:

$$X = \sqrt{\frac{\hbar(S_{BH} - S_0(Q))}{\pi k_B}} \cos(2\pi P_M k_B T_H / \hbar) \quad (13)$$

$$\Pi_X = \sqrt{\frac{\hbar(S_{BH} - S_0(Q))}{\pi k_B}} \sin(2\pi P_M k_B T_H / \hbar) \quad (14)$$

$$Q = Q \quad (15)$$

$$\Pi_Q = P_Q + \Phi P_M + S'_0(Q) P_M T_H / k_B \quad (16)$$

where the ‘entropy at extremality’ is given by $S_0(Q) = \pi k_B Q^2 / \hbar c$, $' \equiv d/dQ$ and Φ is the electrostatic potential at the horizon. The new phase space is \mathcal{R}^4 , on which, a rigorous quantization can be performed in a straightforward fashion. Moreover, as shown in [15] it is straightforward to calculate the adiabatic invariant for charged black holes in this parameterization. In particular, Eqs.(9), (10) generalise to the following adiabatic invariant :

$$\mathcal{J} \equiv \frac{1}{2\pi} \oint \Pi_X dX = \hbar(S_{BH} - S_0(Q)) / 2\pi k_B = (2n + 1)\hbar \quad (17)$$

Quantization yields the following spectra for entropy and charge of the four dimensional quantum black hole (we refer the reader to [15] for details. For relation of the following spectra with that proposed by Bekenstein and collaborators [18,19], we refer to [16]) :

$$S_{BH} = (2n + p + 1) \pi k_B \quad , \quad n = 0, 1, 2, \dots \quad , \quad (18)$$

$$\frac{Q}{e} = m \quad , \quad m = 0, \pm 1, \pm 2, \dots \quad (19)$$

$$p \equiv \frac{Q^2}{\hbar c} = m^2 \alpha \quad . \quad (20)$$

where consistency requires p to be a non-negative integer. This in turn implies that the fine structure constant is constrained to be a rational number:

$$\alpha = \frac{e^2}{\hbar c} = \frac{p}{m^2} \quad , \quad (21)$$

This somewhat strange constraint can be interpreted in either of two ways: (a) given an observed value of α , the black hole quantum numbers p and m must be such that it satisfies this constraint *or* (b) even the presence of a single black hole in the universe constrains the admissible values of α . We will examine the consequences of this constraint in the next section, but first we look at the effects of a small variation of α on the entropy(18)-(21). Since the quantum number m measures the number of constituent fundamental charged particles making up the black hole, it remains fixed during the time in which α varies. This implies p increases if α increases. Since the quantum number n is associated with the adiabatic invariant \mathcal{J} , it remains fixed. We therefore conclude, via Eq.(18), that the black hole entropy S_{BH} increases monotonically with α , irrespective of the source of the α variation. This conclusion is identical to that found in [12], although we arrived at it via a different route. While the one hand, we assumed a specific model of quantum black holes, on the other hand, we did not have to use any detailed thermodynamic stability analysis.

IV. MINIMUM CHARGE AND MASS

In this section, we show that lower bounds on the mass and charge of black holes follow from requiring the variation of α to be compatible with the constraint (21). Consider again Eq.(21), which implies (with m constant as explained before):

$$\frac{\delta\alpha}{\alpha} = \frac{\delta p}{p} \Rightarrow p = \delta p (\delta\alpha/\alpha)^{-1} . \quad (22)$$

Now, since $\delta\alpha/\alpha \approx 10^{-5}$ and $\delta p \geq 1$, it follows that

$$p \geq 10^5 . \quad (23)$$

In general $p = 10^5 \delta p$, where δp is an integer. This, coupled with the relations $Q = \sqrt{p\hbar c} = \sqrt{p} e/\sqrt{\alpha}$ and $\alpha = 1/137$ further imply the condition:

$$Q \geq 3,000 e\sqrt{\delta p} \quad (24)$$

Thus, the minimum charge of a black hole is given by ($\delta p = 1$):

$$Q_{Min} = 3,000 e \quad (25)$$

and higher charges are square root of integer multiples of the above unit. If one uses the limit set by Bahcall et al [3], namely $\delta\alpha/a < 10^{-4}$ over about 10 Billion years, then it can be easily shown that the bound reduces to $Q_{Min} = 1,000e$.

Also, combining (2), (18) and (19), we can write the mass of the black hole as:

$$\frac{M}{M_{Pl}} = \frac{2n + 2p + 1}{2\sqrt{2n + p + 1}} , \quad (26)$$

where $M_{Pl} = 5 \times 10^{-5}g$ is the Planck mass. Let us consider the astrophysically relevant, ‘small charge regime’, defined by:

$$\frac{Q/\sqrt{\hbar c}}{M/M_{Pl}} \equiv k \ll 1 . \quad (27)$$

Using (20) and (26), this gives:

$$n = \frac{1}{k^2} \left[p\sqrt{1 - k^2} + p(1 - k^2) - \frac{k^2}{2} \right] \approx \frac{2p}{k^2} \gg p . \quad (28)$$

In this case, we get from (26):

$$\frac{M}{M_{Pl}} = \frac{\sqrt{2n + 1}}{2} , \quad (29)$$

This, along with (23) and (28) gives a lower bound on the black hole mass:

$$M_{Min} = 200M_{Pl} . \quad (30)$$

Again, we recover the lower bound on mass found in [10], although unlike there, variation of n has not been used to derive the result, since the latter does not in fact change. Once again, the upper bound set by Bahcall et al [3] would reduce M_{Min} to about $70M_{Pl}$. Also, it may be noted that the lower bounds derived here hold for charged black holes, leading one to believe that the constraints (25) and (30) are indeed robust. We observe from Eqs.(25) and (30) that the minimum p and minimum n both increase with decreasing $\delta\alpha/\alpha$. The situation is similar to one encountered in [10] for the quantum number N . If future observations were to make the bound on the α variation even smaller than current values (as opposed to confirming the results of Webb *et al* [1]), the most natural conclusion might be that $\delta\alpha = 0$, since in this case $\delta p = 0 = \delta n$, amounting to no constraints on p and n themselves.

Finally, let us calculate the work done by a black hole of mass M due to evolution of α . Variation of (26) yields (n remaining fixed due to adiabatic invariance).

$$\frac{\delta M}{M} = \frac{1}{2} \frac{\delta c}{c} + \frac{6n + 2p + 3}{2(2n + 2p + 1)(2n + p + 1)} \delta p \approx \frac{1}{2} \frac{\delta c}{c} + \frac{3p}{4n} \frac{\delta\alpha}{\alpha} \approx \frac{1}{2} \frac{\delta c}{c} \quad [n \gg p], \quad (31)$$

where we have used $M_{Pl} = \sqrt{\hbar c/G} \Rightarrow \delta M_{Pl}/M_{Pl} = \delta c/2c$. Since $|\delta c/c| \leq \delta\alpha/\alpha \approx 10^{-5}$, the work done due to (so far unknown) external source which is responsible for the variation of α is bounded from above by:

$$W = \delta(Mc^2) \leq \frac{\delta\alpha}{\alpha}(Mc^2) = 10^{-5}Mc^2 . \quad (32)$$

Thus, for a solar mass black hole, $W \approx 10^{48} \text{ erg}$, about 10^{33} times the Planck energy.

We have argued that the claimed variations of α should be considered adiabatic with respect to the dynamics of all physical black holes. Within a specific model, we showed that such an adiabatic increase of α predicts an increase of S_{BH} , no matter what the source of the variation. This conclusion follows from the spectra of black hole parameters stated in section (III). It differs somewhat from that in [11], but agrees with that in [12].

However, if the claimed variation of α turns out to be real, the spectra in our model also require minimum values of charge and mass, roughly equal to $3,000e$ and $200M_{Pl}$ respectively. Although this may seem somewhat disturbing at first, note that these numbers are by far much less than those associated with astrophysical black holes.

There remain, of course many open questions. For example, are there other theoretical tests which depend only on variation of c or e , and not both? These could be used to determine the source of α variation [29]. Is it legitimate to assume \hbar does not change? What are the implications of a minimum charge and mass in the spectrum for Hawking radiation? Recall that the minimum charge and mass were a direct consequence of the constraint on the fine structure constant in our model. Do similar constraints occur in other models of quantum black holes, or can they be avoided? Finally it would be interesting to repeat the above analysis in the context of the spectrum obtained recently by Medved and Gour for Kerr-Newmann black holes [30].

We hope that answers to at least some of these questions will emerge in the near future as more research gets underway in such a fundamental aspect of experimental and theoretical physics as this. Clearly, if the astrophysical observations about α variation are confirmed by more precision experiments, it may provide a much needed experimental laboratory for the study of some features of quantum gravity.

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